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The balanced-projective dimension of units in commutative modular group algebras

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Abstract

Suppose F is a perfect field of characteristic $p \neq 0$ and G is a multiplicatively written abelian p-group. Write $\mathrm{bpd}(H)$ for the balanced-projective dimension of an arbitrary p-group H. If $V(G)$ is the group of normalized units of the group algebra $F(G)$, it is shown that $\mathrm{bpd}(V(G)) = \mathrm{bpd}(G)$. This was known previously only in the special case where one of the dimensions is zero. Also, some partial results are obtained concerning the conjecture that the functor $G \mapsto V(G)/G$ decreases balanced-projective dimension. Special cases of these results are related to the unresolved direct factor problem: When is G a direct factor of the group of units of $F(G)$? © 1998 Elsevier Science B.V. All rights reserved.

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0. Introduction

For a fixed prime p, let G be a multiplicatively written abelian p-group and let F be any perfect field of characteristic p. If one prefers, F can always be taken to be the prime field of characteristic p without seriously violating the spirit of the results herein; however, once F is chosen, we consider it fixed, whereas the group G is considered a variable. Let $\mathrm{Ab}(p)$ denote the category of (multiplicatively written) abelian p-groups.

The group algebra of G over F is denoted by $F(G)$. The group of units of $F(G)$ is designated simply as $U(G)$, and we let $V(G)$ denote the group of units of augmentation 1. Hence, $V(G)$ is a p-group and $U(G) = V(G) \times F^*$, where F^* is the

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multiplicative group of F. We consider $V: G \mapsto V(G)$ as a functor from Ab(p) to itself. Our main result can be stated as follows.

Theorem 0.1. *The functor V preserves balanced-projective dimension.*

With this objective in mind, we will attempt now to explain all the relevant concepts. At the same time, the connection of Theorem 0.1 with certain results known previously will become clear.

Suppose that $G \in \mathrm{Ab}(p)$. The subgroup of G consisting of all those elements that have pth roots in G is denoted by G^1, while G^0 stands for G itself. For each ordinal α, we define G^α inductively by $G^{\alpha+1} = (G^\alpha)^1$ and $G^\alpha = \bigcap_{\beta<\alpha} G^\beta$ when α is a limit. Recall that a subgroup H of G is said to be *isotype* if $G^\alpha \cap H = H^\alpha$ for each α. Dually, H is *nice* if $(G/H)^\alpha = G^\alpha H/H$. If H is both isotype and nice in G, it is called a *balanced* subgroup.

An important subclass of Ab(p) is the class of groups usually known as the totally projective groups. The structure of one of these groups is completely determined by its Ulm invariants, and the class of totally projective groups has several alternate descriptions other than the one associated with total projectivity. These alternate descriptions include: simply presented, Axiom 3, and balanced-projective [1]. The latter two are the ones relevant for this paper.

Every G in Ab(p) has a balanced-projective resolution; that is, there is an exact sequence

$$(E) \qquad \cdots \longrightarrow A_{k+1} \xrightarrow{\varphi_{k+1}} A_k \longrightarrow \cdots \longrightarrow A_1 \xrightarrow{\varphi_1} A_0 \longrightarrow G \longrightarrow 1$$

where the A_k's are totally projective and $\varphi_{k+1}(A_{k+1})$ is a balanced subgroup of A_k for all $k \geq 0$. The balanced-projective dimension of G, denoted hereafter by bpd(G), is finite if we can choose (E) so that $A_{k+1} = 1$ for some k. In this case, the smallest such k is the balanced-projective dimension of G; that is, $\mathrm{bpd}(G) = k$. If no such k exists, define $\mathrm{bpd}(G) = \infty$ with the convention $\infty > k$ for all integers k. (An obvious version of Schanuel's lemma guarantees that bpd(G) is well defined.) Thus, the totally projective groups are precisely those groups in Ab(p) of balanced-projective dimension 0. Therefore, Theorem 0.1 is a generalization of the fundamental result of May [7] that states that $V(G)$ is totally projective if and only if G is; in particular, May's result establishes Theorem 0.1 for the case $k = 0$.

Another functor closely related to V is $\overline{V}: \mathrm{Ab}(p) \to \mathrm{Ab}(p)$ defined by $\overline{V}(G) = V(G)/G$ for all $G \in \mathrm{Ab}(p)$. There are several results that verify the following conjecture in various special cases [5–8, 10].

Conjecture 0.2. The functor $\overline{V}$ decreases finite nonzero balanced-projective dimension; that is, if $\mathrm{bpd}(G) = k$ for some positive integer k, then $\mathrm{bpd}(\overline{V}(G)) \leq k - 1$.

We are able to verify Conjecture 0.2 in two new cases: when $|G| \leq \aleph_k$, and when G is an isotype subgroup of a totally projective group of length not exceeding ω_k.

The remainder of the paper is divided into two sections. In the next section, we prove Theorem 0.1. In Section 2, we consider Conjecture 0.2.

1. The balanced-projective dimension of $V(G)$

In [7], May showed that if G is a p-group, then G is totally projective if and only if $V(G)$ is totally projective. In the main result of this section (Theorem 1.9), we prove that $\mathrm{bpd}(V(G)) = \mathrm{bpd}(G)$ for all p-groups G. As mentioned previously, our result can be viewed as a generalization to arbitrary balanced-projective dimension of May's result for dimension 0.

Denote the *p-height* of $g \in G$ by $|g|_G$. Thus, if $g \in G^{\alpha} \backslash G^{\alpha+1}$, $|g|_G = \alpha$, while if $g \in G^{\alpha}$ for all ordinals α, we define $|g|_G = \infty$ with the understanding that $\infty > \alpha$ for all α. If κ is a cardinal, a subgroup H of G is *κ-separable* in G if for each $g \in G$ there is a subset $S \subseteq H$ such that $|S| \leq \kappa$ and

$$\sup\{|gh|_G\colon\ h \in H\} = \sup\{|gs|_G\colon\ s \in S\}.$$

Observe that if $\kappa = \aleph_{-1}$ (i.e. if κ is finite), a κ-separable subgroup is simply a nice subgroup. On the other hand, the $\aleph_0$-separable subgroups of G are the original separable subgroups introduced in [3]. The relevance of κ-separable subgroups is revealed by the following definition and theorem, which appear in [2].

Definition 1.1. Let G be a p-group and suppose κ is a cardinal. By an *$H(\kappa)$-family* of G is meant a collection $\mathscr{C}$ of subgroups of G such that

H1. $1 \in \mathscr{C}$.

H2. $\mathscr{C}$ is closed under group union; that is, $\langle N_i\colon\ i \in I\rangle \in \mathscr{C}$ if $N_i \in \mathscr{C}$ for each i.

H3. If A is a subgroup of G of cardinality not exceeding κ, then there is a $N \in \mathscr{C}$ such that $A \subseteq N$ and $|N| \leq \kappa$.

If κ is infinite, G is said to satisfy *Axiom* $3:\kappa$ when G has an $H(\kappa)$-family of κ-separable subgroups. If $\kappa = \aleph_{-1}$ is finite, G satisfies Axiom $3:\kappa$ provided that G has an $H(\aleph_0)$-family of κ-separable (= nice) subgroups. Thus, G satisfies Axiom $3:\aleph_{-1}$ if and only if G is totally projective.

Theorem 1.2 (Fuchs and Hill [2]). *For a p-group G and a nonnegative integer k, the following statements are equivalent*:

(a) $\mathrm{bpd}(G) \leq k$.

(b) *G satisfies Axiom* $3:\aleph_{k-1}$.

(c) *G has a composition series of $\aleph_{k-1}$-separable subgroups; that is, there is a smooth chain*

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_{\alpha} \subseteq \cdots \quad (\alpha < \mu)$$

of $\aleph_{k-1}$-separable subgroups of $G = \bigcup_{\alpha<\mu} N_{\alpha}$ such that $|N_{\alpha+1}/N_{\alpha}|$ is finite whenever $\alpha + 1 < \mu$.

We begin our work with a group-theoretic result, which demonstrates that under certain conditions κ-separability is inductive and transitive. Here, and in the sequel, G always denotes an element of Ab(p) and κ is an arbitrary cardinal (with $\kappa = \aleph_{-1}$ allowed).

Lemma 1.3. *Suppose* $1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots$ $(\alpha < \mu)$ *is a smooth chain of subgroups of* $G = \bigcup_{\alpha<\mu} N_\alpha$ *such that for each* $\alpha < \mu$ *and* $x \in N_{\alpha+1} \backslash N_\alpha$, *there exists a subset* $S_\alpha \subseteq N_\alpha$ *with* $|S_\alpha| \le \kappa$ *and*

$$\sup\{|xy|_G\colon\ y \in N_\alpha\} = \sup\{|xy_\alpha|_G\colon\ y_\alpha \in S_\alpha\}.$$

Then, N_α *is* κ*-separable in* G *for all* α.

Proof. Suppose to the contrary that N_α is not κ-separable for some α. Select an ordinal $\gamma < \mu$ minimal with respect to the property that there exists $x \in N_\gamma$ so that for every subset $S_\alpha \subseteq N_\alpha$ with $|S_\alpha| \le \kappa$, there exists $n_0 \in N_\alpha$ with $|xn_0|_G > |xy|_G$ for all $y \in S_\alpha$. Since the chain of N_α's is smooth, $\gamma = \beta + 1$ for some $\beta > \alpha$.

By hypothesis, there exists $S_\beta \subseteq N_\beta$ such that $|S_\beta| \le \kappa$ and for each $y \in N_\beta$ (and, in particular, for each $y \in N_\alpha$) there is a $y_\beta \in S_\beta$ with $|xy_\beta|_G \ge |xy|_G$. By the minimality of γ and the fact that $|S_\beta| \le \kappa$, we obtain $S_\alpha \subseteq N_\alpha$ such that $|S_\alpha| \le \kappa$ and for each $y_\beta \in S_\beta$ and $n \in N_\alpha$ there exists $n_\alpha \in S_\alpha$ such that $|y_\beta n_\alpha^{-1}|_G \ge |y_\beta n|_G$.

Now select $n_0 \in N_\alpha$ such that $|xn_0|_G > |xn_\alpha|_G$ for all $n_\alpha \in S_\alpha$. Also select $y_\beta \in S_\beta$ such that $|xy_\beta|_G \ge |xn_0|_G$. Then, for all $n_\alpha \in S_\alpha$,

$$|xn_\alpha|_G < |xn_0|_G \le |(xy_\beta)(xn_0)^{-1}|_G = |y_\beta n_0^{-1}|_G \le |y_\beta n^{-1}|_G$$

for some $n \in S_\alpha$. In particular, $|xn|_G < |y_\beta n^{-1}|_G$. Note further that $|xn|_G < |xy_\beta|_G$. Therefore, the contradiction

$$|xn|_G = |xny_\beta n^{-1}|_G = |xy_\beta|_G > |xn|_G.$$

is obtained. $\square$

At this juncture, it is convenient to review some standard notions associated with group algebras. For $v \in F(G)$, we write supp v for the *support* of v and aug v for the *augmentation* of v. Thus, supp v is the collection of those elements of G appearing nontrivially in v, and aug v is the sum of the coefficients of v. Therefore, since the characteristic of F is p and G is a p-group, $V(G) = \{v \in F(G)\colon \text{aug}\, v = 1\}$. An important consequence of our assumption that the field F is perfect is the identity

$$|v|_{V(G)} = \min\{|g|_G\colon\ g \in \operatorname{supp} v\}$$

for all $v \in V(G)$. Finally, if H is a subgroup of G, $K_G(H)$ denotes the kernel of the natural map $V(G) \twoheadrightarrow V(G/H)$ induced by the quotient map $G \twoheadrightarrow G/H$. Observe that if $v \in K_G(H)$, then $H \cap \operatorname{supp} v \ne \emptyset$.

Lemma 1.4. *If* H *is a* κ*-separable subgroup of* G, *then* $K_G(H)$ *is* κ*-separable in* $V(G)$.

Proof. Suppose $vK_G(H)$ is an arbitrary but fixed coset of $K_G(H)$ in $V(G)$ with $v \in V(G)\backslash K_G(H)$. Write v as

$$v = E_1g_1 + E_2g_2 + \cdots + E_ng_n$$

where $E_i \in F(H)$ $(1 \le i \le n)$ and $g_1, g_2, \ldots, g_n \in G$ are distinct modulo H. Observe that the representative v of the coset $vK_G(H)$ can be chosen so that $\operatorname{aug} E_i = e_i \ne 0$ for all i.

Now suppose that w is an arbitrary element of the coset $vK_G(H)$ and write

$$w = D_1g_1 + D_2g_2 + \cdots + D_ng_n + D_{n+1}g_{n+1} + \cdots + D_mg_m$$

where $D_i \in F(H)$ $(1 \le i \le m)$, $g_1, g_2, \ldots, g_n$ are as above, and $g_1, g_2, \ldots, g_n$ together with $g_{n+1}, \ldots, g_m \in G$ are distinct modulo H. Since $wK_G(H) = vK_G(H)$, we conclude that $\operatorname{aug} D_i = e_i$ if $1 \le i \le n$ and $\operatorname{aug} D_{n+1} = \cdots = \operatorname{aug} D_m = 0$. Thus, if we truncate w to form

$$w_t = D_1g_1 + D_2g_2 + \cdots + D_ng_n,$$

then $w_t \in vK_G(H)$ and $|w|_{V(G)} \le |w_t|_{V(G)}$.

For each i $(1 \le i \le n)$, select a subset T_i of H such that $|T_i| \le \kappa$ and

$$\sup\{|g_ih|_G\colon h \in H\} = \sup\{|g_ix|_G\colon x \in T_i\}.$$

Set $T = \bigcup_{1 \le i \le n} T_i$ and define

$$S = \{e_1t_1g_1 + e_2t_2g_2 + \cdots + e_nt_ng_n : t_1, t_2, \ldots, t_n \in T\}.$$

Observe that $S \subseteq vK_G(H)$ and $|S| \le \kappa$. Moreover, for each $w \in vK_G(H)$, there exists $s \in S$ such that $|w|_{V(G)} \le |w_t|_{V(G)} \le |s|_{V(G)}$ and the result follows. $\square$

We remark that the converse of Lemma 1.4 is also true; however, this fact will not be needed. The following technical lemma will be used only in the proof of Proposition 1.7.

Lemma 1.5. *Suppose A is a κ-separable subgroup of G and B is a subgroup of G with $A \subseteq B$ and B/A finite. Let N be a subgroup of $V(G)$ with $K_G(A) \subseteq N$. Suppose further that $v \in V(G)$ and for each $w \in N$ there exists $z_w \in V(B)$ such that $vN = z_wN$ and $|vw|_{V(G)} \le |z_w|_{V(G)}$. Then, if $v \notin N$, there exists $S \subseteq N$ such that $|S| \le \kappa$ and*

$$\sup\{|vw|_{V(G)}\colon w \in N\} = \sup\{|vs|_{V(G)}\colon s \in S\}.$$

Proof. Select and fix a set of representatives $b_1, b_2, \ldots, b_n \in B$ for the distinct cosets of A in B. Since A is κ-separable, there exists a subset $T \subseteq A$ such that $|T| \le \kappa$ and

$$\sup\{|b_ia|_G\colon a \in A\} = \sup\{|b_it|_G\colon t \in T\}$$

for all $i \in \{1,2,\dots,n\}$. For each $w \in N$, write z_w as

$$z_w = E_{1,w}b_1 + E_{2,w}b_2 + \cdots + E_{n,w}b_n$$

where $E_{i,w} \in F(A)$ whenever $1 \le i \le n$. (Observe that this can be done since $\operatorname{supp} z_w \subseteq B$ for all w.) If I is a nonempty subset of $\{1,2,\dots,n\}$, define a subset N_I of N by decreeing that $w \in N_I$ if and only if $\operatorname{aug} E_{i,w} \ne 0$ whenever $i \in I$ and $\operatorname{aug} E_{i,w} = 0$ whenever $i \notin I$. Thus, N is the disjoint union of the finitely many nonempty N_I's.

We claim that for each nonempty $I \subseteq \{1,2,\dots,n\}$ with $N_I \ne \emptyset$, there exists a subset $S_I \subseteq N$ such that $|S_I| \le \kappa$ and

$$\sup\{|vw|_{V(G)}:\ w \in N_I\} = \sup\{|vs|_{V(G)}:\ s \in S_I\}.$$

Once this claim is established, we simply set

$$S = \bigcup\{S_I:\ \emptyset \ne I \subseteq \{1,2,\dots,n\} \text{ and } N_I \ne \emptyset\}$$

to complete the proof of the lemma.

To establish the claim, suppose I is an arbitrary but fixed nonempty subset of $\{1,2,\dots,n\}$ with $N_I \ne \emptyset$. By reindexing if necessary, we may assume $I = \{1,\dots,m\}$ for some $m \le n$. Choose a particular $w_I \in N_I$. For $1 \le i \le m$, set $e_i = \operatorname{aug} E_{i,w_I}$ and define

$$T_I = \{e_1b_1t_1 + \cdots + e_mb_mt_m : t_1,\dots,t_m \in T\}.$$

Recall that $e_i \ne 0$, and observe $T_I \subseteq V(B)$ and $|T_I| \le \kappa$. Moreover, for each $x \in T_I$, $xK_G(A) = z_{w_I}K_G(A)$. Therefore, using the facts that $K_G(A) \subseteq N$ and $z_wN = vN$ for all $w \in N$, $xN = vN$ for all $x \in T_I$.

Now suppose $w \in N_I$, and for convenience of notation set $E_i = E_{i,w}$. Then,

$$z_w = E_1b_1 + \cdots + E_mb_m + E_{m+1}b_{m+1} + \cdots + E_nb_n$$

where $E_1,\dots,E_n \in F(A)$. Recall that $\operatorname{aug} E_i \ne 0$ for $1 \le i \le m$, and $\operatorname{aug} E_i = 0$ for all $i > m$. Thus, if we truncate z_w to form

$$z_{w,t} = E_1b_1 + \cdots + E_mb_m,$$

then $\operatorname{supp} z_{w,t} \subseteq \operatorname{supp} z_w$ implies $|z_{w,t}|_{V(G)} \ge |z_w|_{V(G)}$. For each i, select $t_i \in T$ such that $|b_it_i|_G \ge |g|_G$ for all $g \in \operatorname{supp} E_ib_i$. Then

$$x_w = e_1b_1t_1 + \cdots + e_mb_mt_m \in T_I$$

and $|x_w|_{V(G)} \ge |z_{w,t}|_{V(G)}$.

Now define $S_I = \{xv^{-1}:\ x \in T_I\}$. Recall $xN = vN$ for all x in T_I so that $S_I \subseteq N$. Also, $|T_I| \le \kappa$ implies $|S_I| \le \kappa$. Moreover, for all $w \in N_I$,

$$|vw|_{V(G)} \le |z_w|_{V(G)} \le |z_{w,t}|_{V(G)} \le |x_w|_{V(G)} = |v(x_wv^{-1})|_{V(G)}.$$

Since $x_w \in T_I$, $x_wv^{-1} \in S_I$ and the claim is established. □

If A is a subgroup of G, recall that $I(A)=\{v\in F(A)\colon \operatorname{aug} v=0\}$ is the ideal of $F(A)$ generated by $\{a-1\colon a\in A\}$. Moreover, $K_G(A)=1+F(G)\cdot I(A)$, where $F(G)\cdot I(A)$ is the ideal of $F(G)$ generated by $I(A)$, and hence by $\{a-1\colon a\in A\}$ as well. Further observe that $F(G)\cdot I(A)$ is the kernel of the ring-homomorphism $F(G)\twoheadrightarrow F(G/A)$ induced by the quotient map $G\twoheadrightarrow G/A$.

Lemma 1.6. *Suppose A and B are subgroups of G with $A\subseteq B$ and B/A finite. Then there exists an integer $n\geq 1$ such that $I(B)^n\subseteq F(B)\cdot I(A)$, where $I(B)^n$ is the ideal product in $F(B)$ of n copies of $I(B)$.*

Proof. Suppose $|B/A|=p^m$ and select n large enough so that any product of the form

$$(b_1-1)(b_2-1)\cdots(b_n-1)\quad (b_i\in B,\ 1\leq i\leq n) \tag{1}$$

involves at least p^m of the b_i's that represent the same coset of A in B. (Observe that $n=p^m(p^m-1)+1$ would suffice.)

Now suppose

$$v=(b_1-1)(b_2-1)\cdots(b_n-1)$$

is of the form (1), and reindex as necessary so that $b_iA=b_jA$ whenever $1\leq i\leq j\leq p^m$. If $\varphi: F(B)\twoheadrightarrow F(B/A)$ is the ring-homomorphism induced by the quotient map $B\twoheadrightarrow B/A$, then $(bA-1A)^{p^m}=b^{p^m}A-1A=0$ for all $b\in B$ implies

$$\varphi(v)=(b_1A-1A)^{p^m}\prod_{i>p^m}(b_iA-1A)=0.$$

Hence, $v\in \operatorname{Ker}\varphi=F(B)\cdot I(A)$. Since the collection of all elements of the form (1) is a set of generators of $I(B)^n$, we conclude that $I(B)^n\subseteq F(B)\cdot I(A)$. □

In preparation for our next result, we now set some additional notation. Let A and B be subgroups of G with $A\subseteq B$ and B/A finite. Select a set of representatives R_A for the distinct cosets of A in G with $1\in R_A$. Since $A\subseteq B$, there is a set of representatives R_B for the distinct cosets of B in G such that $R_B\subseteq R_A$ and $1\in R_B$. Select ordinals $\lambda\geq\mu$ so that $R_A=\{g_\alpha\colon \alpha<\lambda\}$ and $R_B=\{g_\alpha\colon \alpha<\mu\}$ with $g_0=1$. For each integer $r\geq 1$ and ordinal $\sigma\leq\mu$, define

$$K_{r,\sigma}=1+\sum_{\alpha<\lambda}I(A)g_\alpha+\sum_{0\leq\alpha<\sigma}I(B)^r g_\alpha+\sum_{\sigma\leq\alpha<\mu}I(B)^{r+1}g_\alpha. \tag{$*$}$$

Here, for example, $\sum_{\alpha<\lambda}I(A)g_\alpha$ represents the set of all sums $\sum_{\alpha<\lambda}D_\alpha g_\alpha$ with $D_\alpha\in I(A)$ and $D_\alpha=0$ for almost all α. Since $I(A)\subseteq I(B)$ and $I(A)\cdot I(B)^r\subseteq I(B)^{r+1}$, it is clear that each $K_{r,\sigma}$ is closed under multiplication. Moreover, $K_{r,\sigma}\subseteq K_G(B)$, a p-group. Therefore, $K_{r,\sigma}$ is a subgroup of $V(G)$ with $K_G(A)\subseteq K_{r,\sigma}\subseteq K_G(B)$. We remark that constructions similar to ($*$) were also used in [7] and [9]. In fact, the construction in [7] coincides with ours in the special case $A=1$ and B finite.

By Lemma 1.6, there exists an integer $n \geq 1$ such that

$$I(B)^n \subseteq F(B) \cdot I(A) \subseteq F(G) \cdot I(A) = \sum_{\alpha<\lambda} I(A)g_\alpha.$$

Thus, $K_{n,\sigma} = K_{n,\mu} = K_G(A)$ for all $\sigma \leq \mu$. Observe further that $K_{1,\mu} = K_G(B)$ and $K_{r+1,\mu} = K_{r,0}$ whenever $1 \leq r < n$. Finally, as in the specialized case of [7], it is seen that the $K_{r,\sigma}$'s form a smooth chain of subgroups from $K_G(A)$ to $K_G(B)$, where the chain is ordered lexicographically by the pairs $(-r, \sigma)$. With the above notation in force, we have the following.

Proposition 1.7. *Suppose A is a κ-separable subgroup of G, and B is a subgroup of G with $A \subseteq B$ and B/A finite. Suppose further that N is a subgroup of $V(G)$ with $K_{r,\sigma} \subseteq N \subseteq K_{r,\sigma+1}$ for some integer $r \geq 1$ and ordinal $\sigma < \mu$. Then, for each $v \in K_{r,\sigma+1} \setminus N$, there exists a subset $S \subseteq N$ such that $|S| \leq \kappa$ and*

$$\sup\{|vw|_{V(G)}\colon w \in N\} = \sup\{|vs|_{V(G)}\colon s \in S\}.$$

Proof. Fix $v \in K_{r,\sigma+1} \setminus N$ and suppose $w \in N$. Then $vw \in K_{r,\sigma+1}$ and we have

$$vw = 1 + \sum_{\alpha<\lambda} D_\alpha g_\alpha + \sum_{\alpha\leq\sigma} E_\alpha g_\alpha + \sum_{\sigma<\alpha<\mu} E_\alpha g_\alpha$$

where $D_\alpha \in I(A)$, $E_\alpha \in I(B)^r$ if $\alpha \leq \sigma$, $E_\alpha \in I(B)^{r+1}$ if $\sigma < \alpha < \mu$, and almost all D_α's and E_α's are 0. Set $E(w) = D_\sigma g_\sigma + E_\sigma g_\sigma + u$ where

$$u = \sum \{D_\alpha g_\alpha\colon \sigma < \alpha < \lambda \text{ and } Bg_\alpha = Bg_\sigma\}.$$

Observe that $\operatorname{supp}(E(w)) \subseteq Bg_\sigma$ and $\emptyset \neq \operatorname{supp}(E(w)) \subseteq \operatorname{supp}(vw-1)$ since $vw \in K_{r,\sigma+1} \setminus K_{r,\sigma}$. Thus, $\langle B, g_\sigma\rangle$ is a finite extension of A containing $\operatorname{supp}(1+E(w))$ and $|vw|_{V(G)} \leq |1+E(w)|_{V(G)}$ for all $w \in N$. We conclude that the result will follow from Lemma 1.5 (with $z_w = 1 + E(w)$) once we have shown $vN = (1 + E(w))N$.

For convenience of notation, temporarily set $E = E(w)$. Define $\delta_0 = vw - 1$ and $\delta_k = \delta_{k-1}(\delta_{k-1} - E)$ for all $k \geq 1$. A routine induction shows that for every $k \geq 0$,

$$\delta_k \in \sum_{\alpha<\lambda} I(A)g_\alpha + \sum_{\alpha<\mu} I(B)^{r+k} g_\alpha.$$

Therefore, $1 + E - \delta_0 \in K_{r,\sigma} \subseteq N$, and for $k \geq 1$, $1 + \delta_k \in K_{r,0} \subseteq K_{r,\sigma} \subseteq N$. Then,

$$vN = vwN = (1 + \delta_0)(1 + E - \delta_0)\left(\prod_{k=1}^{n-1}(1 + \delta_k)\right) N = (1 + E - \delta_n)N$$

for all $n \geq 2$. Since B/A is finite, by Lemma 1.6 we can select an integer $n_0 \geq 2$ so that

$$I(B)^{r+n_0} \subseteq F(B) \cdot I(A) \subseteq F(G) \cdot I(A) = \sum_{\alpha<\lambda} I(A)g_\alpha.$$

Consequently, $\delta_{n_0} \in \sum_{\alpha<\lambda} I(A)g_\alpha$. Since $K_G(A) \subseteq N$, we conclude that

$$vN = (1 + E(w) - \delta_{n_0})N = (1 + E(w))N$$

regardless of the choice of $w \in N$. As remarked above, an application of Lemma 1.5 completes the proof. □

Using the terminology of [7], call a subgroup N of $V(G)$ *saturated* if $\operatorname{supp} v \subseteq N$ for all $v \in N$. The next result was established by [7] in the special case $\kappa = \aleph_{-1}$.

Lemma 1.8. *Suppose N is a saturated subgroup of $V(G)$. If N is κ-separable in $V(G)$, then $N \cap G$ is κ-separable in G.*

Proof. Suppose $g \in G \backslash N \cap G$ and select a subset $T \subseteq N$ so that $|T| \leq \kappa$ and

$$\sup\{|gv|_{V(G)}: v \in N\} = \sup\{|gw|_{V(G)}: w \in T\}.$$

Set $S = \{h: h \in \operatorname{supp} w \text{ for some } w \in T\}$. Then, $S \subseteq N \cap G$ and $|S| \leq \kappa$. Moreover, for each $x \in N \cap G$, there exist $w \in T$ and $h \in \operatorname{supp} w$ such that

$$|gx|_G = |gx|_{V(G)} \leq |gw|_{V(G)} \leq |gh|_G.$$

Since $h \in S$, the result follows. □

We now present our main result. Recall the standing hypothesis that the field F of coefficients of $F(G)$ is perfect of characteristic $p \neq 0$.

Theorem 1.9. *For any p-group G,* $\operatorname{bpd}(V(G)) = \operatorname{bpd}(G)$.

Proof. We claim that if k is a nonnegative integer, then $\operatorname{bpd}(G) \leq k$ if and only if $\operatorname{bpd}(V(G)) \leq k$. The case $k = 0$ is the result of [7] mentioned at the beginning of this section. Therefore, we may assume $k \geq 1$.

First suppose that $\operatorname{bpd}(G) \leq k$. Then, by Theorem 1.2, there exists a smooth chain

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

of $\aleph_{k-1}$-separable subgroups of $G = \bigcup_{\alpha<\mu} N_\alpha$ with $N_{\alpha+1}/N_\alpha$ finite for all α. By Lemma 1.4, the chain

$$1 = K_G(N_0) \subseteq K_G(N_1) \subseteq \cdots \subseteq K_G(N_\alpha) \subseteq \cdots \quad (\alpha < \mu) \tag{$*$}$$

is a smooth chain of $\aleph_{k-1}$-separable subgroups of $V(G) = \bigcup_{\alpha<\mu} K_G(N_\alpha)$. Applying Lemma 1.3 and Proposition 1.7, we can refine $(*)$ to a composition series of $\aleph_{k-1}$-separable subgroups of $V(G)$. Therefore, $\operatorname{bpd}(V(G)) \leq k$ by Theorem 1.2.

Conversely, suppose $\operatorname{bpd}(V(G)) \leq k$. Then, by Theorem 1.2, $V(G)$ has an $H(\aleph_{k-1})$-family $\mathscr{C}$ of $\aleph_{k-1}$-separable subgroups. As shown by May [7], the collection $\mathscr{S}$ of all saturated subgroups of $V(G)$ is an $H(\aleph_0)$-family in $V(G)$. It is now routine to verify that $\mathscr{C} \cap \mathscr{S}$ is an $H(\aleph_{k-1})$-family of saturated $\aleph_{k-1}$-separable subgroups of $V(G)$. Thus, we obtain a smooth chain

$$1 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

of saturated $\aleph_{k-1}$-separable subgroups in $\mathscr{C} \cap \mathscr{S}$ with $V(G) = \bigcup_{\alpha<\mu} N_\alpha$ and the cardinality of $N_{\alpha+1}/N_\alpha$ does not exceed $\aleph_{k-1}$ for all $\alpha < \mu$. By Lemma 1.8 the chain

$$1 = N_0 \cap G \subseteq N_1 \cap G \subseteq \cdots \subseteq N_\alpha \cap G \subseteq \cdots \quad (\alpha < \mu) \tag{$**$}$$

consists of $\aleph_{k-1}$-separable subgroups of $G = \bigcup_{\alpha<\mu}(N_\alpha \cap G)$ with

$$|(N_{\alpha+1} \cap G)/(N_\alpha \cap G)| \le \aleph_{k-1}.$$

Observe that if any subgroup of cardinality $\le \aleph_{k-1}$ is adjoined to an $\aleph_{k-1}$-separable subgroup, the resulting subgroup is again $\aleph_{k-1}$-separable. Consequently, the chain $(**)$ can be refined to a composition series of $\aleph_{k-1}$-separable subgroups of G and we have $\mathrm{bpd}(G) \le k$ by Theorem 1.2. Thus, the claim is established; that is, $\mathrm{bpd}(G) \le k$ if and only if $\mathrm{bpd}(V(G)) \le k$ for all nonnegative integers k.

To complete the proof of the theorem, observe that if one of $\mathrm{bpd}(G)$ or $\mathrm{bpd}(V(G))$ is ∞ (respectively 0), the remaining one must also be ∞ (respectively 0). These facts follow easily from what we have shown above. Moreover, if $\mathrm{bpd}(G) = k$ with $1 \le k < \infty$, then $\mathrm{bpd}(V(G)) \le k$. However, if $\mathrm{bpd}(V(G)) \le k-1$, the contradiction $\mathrm{bpd}(G) \le k-1$ is obtained. Therefore, $\mathrm{bpd}(V(G)) = k = \mathrm{bpd}(G)$. □

Corollary 1.10. *If $|G| \le \aleph_k$ for some nonnegative integer k, then* $\mathrm{bpd}(V(G)) \le k$.

Proof. It is well known and easily verified that if $|G| \le \aleph_k$, then $\mathrm{bpd}(G) \le k$. Therefore, the result is an immediate consequence of Theorem 1.9. □

2. The balanced-projective dimension of $V(G)/G$

In this section, the principal results appear below as Theorems 2.5 and 2.8. These are the special cases of Conjecture 0.2 mentioned in the introductory Section 0.

Two subgroups A and B of G are *compatible* in G, written $A \| B$, if for every pair $(a,b) \in A \times B$ there exists $c \in A \cap B$ such that $|ab|_G \le |ac|_G$. Observe that compatibility is a symmetric relation. For a subgroup H of G, define the *height spectrum* of H in G by

$$\mathrm{htspec}_G(H) = \{\alpha\colon\ \alpha = |h|_G \text{ for some } h \in H\}.$$

We now have the following two group-theoretic lemmas.

Lemma 2.1. *Suppose H is a subgroup of G. If for each $g \in G$ there exists a subgroup C of G such that $g \in C$, $C \| H$ in G and $|\mathrm{htspec}_G(C)| \le \kappa$, then H is κ-separable in G.*

Proof. Since $C \| H$, for each $h \in H$ there exists $z \in C \cap H$ such that $|gh|_G \leq |gz|_G$. Thus,

$$\sup\{|gh|_G\colon h \in H\} = \sup\{|gz|_G\colon z \in C \cap H\}.$$

But $|\mathrm{htspec}_G(C)| \leq \kappa$, so there exists $S \subseteq C \cap H$ such that $|S| \leq \kappa$ and $\{|gz|_G\colon z \in C \cap H\} = \{|gs|_G\colon s \in S\}$. Therefore,

$$\sup\{|gh|_G\colon h \in H\} = \sup\{|gs|_G\colon s \in S\}$$

and H is κ-separable in G. $\square$

Lemma 2.2 (Hill and Ullery [6]). *Suppose A and B are subgroups of G and N is a nice subgroup of G with $N \subseteq B$. If $A \| B$ in G, then $(AN/N) \| (B/N)$ in G/N.*

In order to apply Lemmas 2.1 and 2.2 to the group algebra setting, we now collect some needed facts. Part (a) below appears in [5], and parts (b) and (c) follow from results in [7] and [8].

Lemma 2.3 (Hill and Ullery [5] and May [7, 8]). *Suppose H is a subgroup of G. Then*

(a) *$GV(H) \| V(A)$ in $V(G)$ for every subgroup A of G.*
(b) *$GV(H)$ is a nice subgroup of $V(G)$ and is balanced in $V(G)$ if H is isotype in G. In particular, G is balanced in $V(G)$.*
(c) *$V(H)$ is nice (respectively isotype) in $V(G)$ if and only if H is nice (respectively isotype) in G.*

Proposition 2.4. *Suppose H and A are subgroups of G. If $|A| \leq \kappa$, then $GV(H)N/G$ is κ-separable in $V(G)/G$ for every subgroup N of $V(A)$.*

Proof. Suppose $vG \in V(G)/G$ with $v \in V(G) \backslash GV(H)N$. Select a subgroup B of G so that $A \subseteq B$, $|B| \leq \kappa$, and $\mathrm{supp}\, v \subseteq B$. Then $vG \in GV(B)/G$. Moreover, since $GV(H)N \|$ $V(B)$ in $V(G)$ by Lemma 2.3(a) and G is nice in $V(G)$ by Lemma 2.3(b), it follows from Lemma 2.2 that $(GV(H)N/G) \| (GV(B)/G)$ in $V(G)/G$. Therefore, in view of Lemma 2.1, it suffices to show that $|\mathrm{htspec}_{V(G)/G}(GV(B)/G)| \leq \kappa$.

If $wG \in GV(B)/G$ with $w \in V(B) \backslash G$, write

$$w = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$$

where $n \geq 2$, $c_1, c_2, \ldots, c_n$ are nonzero elements of F, and $b_1, b_2, \ldots, b_n$ are distinct elements of B. Observe that for every $g \in G$,

$$|wb_1^{-1}g|_{V(G)} = |c_1 g + c_2 b_2 b_1^{-1} g + \cdots + c_n b_n b_1^{-1} g|_{V(G)} \leq |g|_G.$$

Thus, $|wb_1^{-1}|_{V(G)} \geq |wb_1^{-1}g|_{V(G)}$ for all $g \in G$. Therefore,

$$|wG|_{V(G)/G} = |wb_1^{-1}G|_{V(G)/G} = |wb_1^{-1}|_{V(G)} = |b|_G$$

for some $b \in \operatorname{supp}(wb_1^{-1}) \subseteq B$. Since $|B| \le \kappa$, we conclude that $GV(B)/G$ has height spectrum in $V(G)/G$ of cardinality not exceeding κ. □

Recall that if $|G| \le \aleph_k$, then $\operatorname{bpd}(G) \le k$. Thus, the following is a special case of Conjecture 0.2.

Theorem 2.5. *Suppose G is a p-group and k is a positive integer. If $|G| \le \aleph_k$, then* $\operatorname{bpd}(V(G)/G) \le k-1$.

Proof. Select a smooth chain

$$1 = H_0 \subseteq H_1 \subseteq \cdots \subseteq H_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

of subgroups of $G = \bigcup_{\alpha<\mu} H_\alpha$ so that $|H_\alpha| \le \aleph_{k-1}$ for all α. For each α, select a smooth chain

$$1 = A_{\alpha,0} \subseteq A_{\alpha,1} \subseteq \cdots \subseteq A_{\alpha,\beta} \subseteq \cdots \quad (\beta < \nu(\alpha))$$

of subgroups of $H_{\alpha+1} = \bigcup_{\beta<\nu(\alpha)} A_{\alpha,\beta}$, where the ordinal $\nu(\alpha) \le \omega_{k-1}$ depends on the choice of α, and $|A_{\alpha,\beta}| \le \aleph_{k-2}$ for all α and β. By Proposition 2.4, the chain of subgroups

$$\{GV(H_\alpha)V(A_{\alpha,\beta})/G:\ \alpha<\mu,\ \beta<\nu(\alpha)\}$$

can be refined to a composition series of $\aleph_{k-2}$-separable subgroups of $V(G)/G$. Therefore, $\operatorname{bpd}(V(G)/G) \le k-1$ by Theorem 1.2. □

Theorem 2.5 generalizes Theorem 2.6 of [10], where the result was shown to hold in the special case $k \le 2$. Moreover, the case $k=1$ yields a short proof of the main result of [5].

Corollary 2.6 (Hill and Ullery [5]). *If G is of cardinality not exceeding $\aleph_1$, then G is a direct factor of $V(G)$ and $V(G)/G$ is totally projective.*

Proof. By Theorem 2.5, $\operatorname{bpd}(V(G)/G) = 0$. Therefore, $V(G)/G$ is totally projective. Since G is balanced in $V(G)$ and totally projective groups are balanced-projectives, G is a direct factor of $V(G)$. □

We remark that Corollary 2.6 was also shown by May [8] under the additional hypothesis that the length of G does not exceed ω_1.

In preparation for the proof of our final theorem, we make note of some pertinent facts regarding the kernels $K_G(C)$.

Lemma 2.7 (Hill and Ullery [6]). *Suppose A, B and C are subgroups of G.*

(a) *If $A = B \times C$, then $V(A) = V(B) \times (K_G(C) \cap V(A))$.*

(b) *If $1 \ne v \in K_G(C)$, there exists $c \in C$ such that $c \ne 1$ and $|v|_{V(G)} \le |c|_G$.*

The next result generalizes Theorem 10 of [6]. Recall that the *length* of G, which we sometimes write as length(G), is the smallest ordinal α such that $G^\alpha = G^{\alpha+1}$. For an indication of the importance of the class of isotype subgroups of totally projectives, we refer the reader to [4].

Theorem 2.8. *Suppose H is an isotype subgroup of a reduced totally projective p-group G. If the length of G does not exceed ω_k for some positive integer k, then* $\mathrm{bpd}(V(H)/H) \leq k-1$.

Proof. Without loss we may assume that length$(G) = \omega_k$. Therefore, for some limit ordinal μ, $G = \coprod_{i<\mu} G_i$ where length$(G_i) < \omega_k$ for all $i < \mu$. Set $P_0 = 1$ and if $1 \leq \alpha < \mu$, define $P_\alpha = \coprod_{i<\alpha} G_i$. Thus, we have a smooth chain

$$1 = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_\alpha \subseteq \cdots \quad (\alpha < \mu)$$

of direct factors of $G = \bigcup_{\alpha<\mu} P_\alpha$ with $P_{\alpha+1}/P_\alpha$ totally projective of length $< \omega_k$ for all $\alpha < \mu$. As a consequence, there is a smooth chain

$$GV(H) \subseteq GV(H)V(P_1) \subseteq \cdots \subseteq GV(H)V(P_\alpha) \subseteq \cdots \quad (\alpha < \mu)$$

from $GV(H)$ to $V(G) = \bigcup_{\alpha<\mu} GV(H)V(P_\alpha)$.

We claim that each $GV(H)V(P_\alpha)$ is $\aleph_{k-1}$-separable in $V(G)$. In fact, we show more generally that if W is a nice subgroup of $V(G)$ with $V(P_\alpha) \subseteq W \subseteq V(P_{\alpha+1})$ for some α, then $GV(H)W$ is $\aleph_{k-1}$-separable in $V(G)$. Because W is nice in $V(G)$, it is enough to show that $GV(H)W/W$ is $\aleph_{k-1}$-separable in $V(G)/W$. Suppose now that vW is an arbitrary (but fixed) element of $V(G)/W$ with $v \in V(G)\backslash W$. Since $\operatorname{supp} v$ is finite, there exists a direct factor P of G such that $v \in V(P)$, $P_{\alpha+1} \subseteq P$, and $P/P_{\alpha+1}$ is totally projective of length $< \omega_k$. Observe that $GV(H) \parallel V(P)$ in $V(G)$ by Lemma 2.3(a); so the niceness of W in $V(G)$ together with Lemma 2.2 implies that $(GV(H)W/W) \parallel (V(P)/W)$ in $V(G)/W$. Thus, in view of Lemma 2.1, the claim will be established once we have shown that the height spectrum of $V(P)/W$ in $V(G)/W$ has cardinality not exceeding $\aleph_{k-1}$. Consider the natural map $V(P)/V(P_\alpha) \twoheadrightarrow V(P)/W$ with kernel $W/V(P_\alpha)$ nice in $V(G)/V(P_\alpha)$. Since $V(P)$ is balanced in $V(G)$ by Lemma 2.3(c), $W/V(P_\alpha)$ is nice in $V(P)/V(P_\alpha)$. Also, $V(P)/V(P_\alpha)$ and $V(P)/W$ are isotype in $V(G)/V(P_\alpha)$ and $V(G)/W$, respectively. Therefore, the height spectrum of $V(P)/W$ in $V(G)/W$ is a subset of the height spectrum of $V(P)/V(P_\alpha)$ in $V(G)/V(P_\alpha)$. Thus, to obtain the conclusion that $GV(H)W$ is $\aleph_{k-1}$-separable in $V(G)$, it is enough to show that $V(P)/V(P_\alpha)$ has height spectrum in $V(G)/V(P_\alpha)$ of cardinality not exceeding $\aleph_{k-1}$.

Suppose $wV(P_\alpha) \in V(P)/V(P_\alpha)$ with $w \in V(P)\backslash V(P_\alpha)$. By our choice of P and the construction of the P_α's, $P = P_\alpha \times C$ for some direct factor C of G with length$(C) < \omega_k$. Since $V(P) = V(P_\alpha) \times (V(P) \cap K_G(C))$ by Lemma 2.7(a), we may assume that $w \in V(P) \cap K_G(C)$. Noting that $V(P)$, $V(P_\alpha)$ and $V(P) \cap K_G(C)$ are all direct factors of $V(G)$, we obtain in conjunction with Lemma 2.7(b) that

$$|wV(P_\alpha)|_{V(G)/V(P_\alpha)} = |w|_{V(G)} \leq |c|_G < \omega_k$$

for some nonidentity $c \in C$. It now follows that $|wV(P_\alpha)|_{V(G)/V(P_\alpha)}$ has at most $\aleph_{k-1}$ many possible values as w ranges over $V(P)$; in other words, the height spectrum of $V(P)/V(P_\alpha)$ in $V(G)/V(P_\alpha)$ has cardinality not exceeding $\aleph_{k-1}$. Therefore, the claim is established; that is, for all $\alpha < \mu$, $GV(H)W$ is $\aleph_{k-1}$-separable in $V(G)$ whenever W is a nice subgroup of $V(G)$ with $V(P_\alpha) \subseteq W \subseteq V(P_{\alpha+1})$.

Now observe that $V(P_{\alpha+1})/V(P_\alpha)$ is totally projective, $V(P_\alpha)$ is nice in $V(P_{\alpha+1})$ and $V(P_{\alpha+1})$ is balanced in $V(G)$. Thus, there exists a smooth chain

$$V(P_\alpha) = W_{\alpha,0} \subseteq W_{\alpha,1} \subseteq \cdots \subseteq W_{\alpha,\beta} \subseteq \cdots \quad (\beta < \gamma)$$

of nice subgroups of $V(G)$ with $V(P_{\alpha+1}) = \bigcup_{\beta<\gamma} W_{\alpha,\beta}$, where γ is an ordinal depending on the choice of α and $|W_{\alpha,\beta+1}/W_{\alpha,\beta}|$ is finite. By what we have shown above, the subgroups $GV(H)W_{\alpha,\beta}$ form a smooth chain of $\aleph_{k-1}$-separable subgroups of $V(G)$ from $GV(H)V(P_\alpha)$ to $GV(H)V(P_{\alpha+1})$. Putting these chains together and going modulo $GV(H)$, the fact that $GV(H)$ is nice in $V(G)$ implies that $\{GV(H)W_{\alpha,\beta}/GV(H)\}$ is a composition series of $\aleph_{k-1}$-separable subgroups of $V(G)/GV(H)$. Thus, from Theorem 1.2 we have $\operatorname{bpd}(V(G)/GV(H)) \le k$. Moreover, there is a balanced-exact sequence

$$GV(H)/G \rightarrowtail V(G)/G \twoheadrightarrow V(G)/GV(H)$$

with $V(G)/G$ totally projective. Therefore, $V(H)/H \cong GV(H)/G$ has balanced-projective dimension $\le k-1$. □

By means of Theorem 1.2, it is not difficult to show that if H is an isotype subgroup of a totally projective group of length not exceeding ω_k, then $\operatorname{bpd}(H) \le k$. Thus, like Theorem 2.5, Theorem 2.8 is also a special case of Conjecture 0.2. In conclusion, we observe that the main result of [6] is an easily obtained consequence of Theorem 2.8.

Corollary 2.9 (Hill and Ullery [6]). *Suppose H is an isotype subgroup of a coproduct of countable reduced p-groups. Then, H is a direct factor of $V(H)$ and $V(H)/H$ is totally projective.*

Proof. Suppose G is a coproduct of countable reduced p-groups which contains H as an isotype subgroup. Then, G is totally projective of length not exceeding ω_1. Thus, by Theorem 2.8, $\operatorname{bpd}(V(H)/H) = 0$ so that $V(H)/H$ is totally projective and hence balanced-projective. Since H is balanced in $V(H)$, H is a direct factor of $V(H)$. □

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